Mathematics - Course 421

ALGEBRA FUNDAMENTALS

I Introduction

Basic operations in algebra are the same as they are in arithmetic, except that letters are used to stand for numbers. This gives the advantage that one can manipulate numbers without knowing their values. As will be seen in lesson 421.20-2, this advantage is useful in setting up and solving proportions, manipulating formulas, and solving problems in one unknown.

II Evaluation of Algebraic Expressions by Substitution

To evaluate an algebraic expression by substitution, substitute the given numerical values for the variables (letters), and then simplify using "BEDMAS" for correct order of operations (cf lesson 421.10-1, section V).

Example 1:

Evaluate $a + 3b$ if $a = 5$ and $b = -2$.

Solution:

$$
\begin{aligned}
a + 3b &= 5 + 3(-2) && \text{(substitute)} \\
&= 5 + (-6) && \text{(x precedes +)} \\
&= -1
\end{aligned}
$$

Example 2:

Evaluate $(x + y) \div (x)(y)$ if $x = 7$, $y = -4$.

Solution:

$$
\begin{aligned}
(x + y) \div (x)(y) &= (7 + (-4)] \div (7)\ (-4) && \text{(substitute)} \\
&= (3) \div 7(-4) && \text{(brackets first)} \\
&= (\tfrac{3}{7})(-4) && \text{(}\div\text{, x as they occur)} \\
&= (\tfrac{3}{7})(\tfrac{-4}{1}) \\
&= \frac{-12}{7} \\
&= -1\tfrac{5}{7}
\end{aligned}
$$

February 1979

III Powers

a) Notation

Recall that a^n stands for n factors of a:

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \; a}_{\text{n factors of a}}$$

eg, $\qquad x^5 = x \cdot x \cdot x \cdot x \cdot x$

Note the use of the dot to indicate multiplication, in order to avoid confusion of the times sign "x" with the letter "x". Sometimes brackets are used to indicate multiplication.

eg, $\qquad$ $3x(-y)$ means 3 times x times $-y$, but
$3x-y$ means 3 times x, subtract y.

Most often, however, when variables are multiplying each other, the sign is omitted altogether.

eg, $\qquad$ $-3xy$ means -3 times x times y.

A *power* consists of a *base* and an *exponent:*

$$a^n \qquad \left.\begin{array}{l} n \longleftarrow \text{exponent} \\ a \longleftarrow \text{base} \end{array}\right\} \text{power}$$

** NB *Exponentiation* (raising a base to an exponent) takes precedence over multiplication and division.

eg, $\qquad xy^2 = xyy \qquad$ (y must be squared before multiplying by x)

But this natural order of precedence can be overruled with the use of brackets:

eg, $xy^2 = xyy$, but $(xy)^2 = (xy)(xy)$

eg, $-2x^2 = -2xx$, but $(-2x)^2 = (-2x)(-2x)$

eg, $-10^2 = -(10)(10)$, but $(-10)^2 = (-10)(-10)$

b) Power Laws

Nine basic laws governing operations with exponents follow. A brief rationale and one or more examples are included with each law.

Law 1:

$$\boxed{x^n \cdot x^m = x^{m + n}}$$

Rationale:

(n factors of x)(m factos of x) = (m + n) factors of x

Example:

$$x^5 \cdot x^7 = x^{12}$$

Law 2:

$$\boxed{\left.\begin{array}{l} x^n \div x^m \\ \text{or } \dfrac{x^n}{x^m} \end{array}\right\} = x^{n - m}}$$

Rationale:

a) If $n > m$, cancelling m common factors of x leaves $n - m$ factors of x in the numerator.

b) If $n < m$, cancelling n common factors of x leaves $m - n$ factors of x in the donominator.

ie, $\dfrac{1}{x^{m - n}} = x^{-(m - n)} = x^{n - m}$ (cf law 4)

Examples:

$$x^7 \div x^5 = x^{7-5} = x^2$$

$$x^5 \div x^7 = x^{5-7} = x^{-2}$$

amounts to cancelling 5 factors of x in either case

** NB In laws 1 and 2, the bases of the powers must be identical

ie, $2^5 \times 2^7 = 2^{12}$, but $2^5 \times 3^7$ cannot be simplified as a power

Similarly, $\dfrac{2^7}{2^5} = 2^2$, but $\dfrac{2^7}{3^5}$ cannot be simplified as a power

Law 3:

$$\boxed{(x^n)^m = x^{mn}}$$

Rationale:

m factors of (n factors of x) = mn factors of x

Example:

$$(x^7)^5 = x^{35}$$

Law 4:

$$\boxed{\begin{array}{l} x^{-m} = \dfrac{1}{x^m} \\ \text{or} \quad \dfrac{1}{x^{-m}} = x^m \end{array}}$$

Rationale:

Negative exponents are defined this way to make the other laws consistent.

eg, $\frac{a^3}{a^5} = \frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a} = \frac{1}{a^2}$

But law 2 gives $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$

These are consistent only if $\frac{1}{a^2} = a^{-2}$

<u>Examples</u>:

$$x^{-5} = \frac{1}{x^5}$$

$$\frac{1}{x^{-5}} = x^5$$

Thus powers may be shifted from numerator to denominator, and vice versa, merely by changing the sign of their exponents.

<u>Law 5</u>:

$$\boxed{x^0 = 1}$$

Rationale:

$$\frac{x^n}{x^n} = x^{n-n} = x^0 \text{ by law 2}$$

But $\frac{x^n}{x^n} = 1$ by cancelling numerator and denominator

$\therefore x^0 = 1$ to make answers consistent

<u>Eamples</u>:

$$10^0 = 1$$

$$(-13)^0 = 1$$

$$(xy)^0 = 1$$

Law 6:

$$\boxed{(xy)^m = x^m y^m}$$

Rationale:

m factors of xy = (m factors of x)(m factors of y) just by reordering the x's and y's.

Examples:

$$(xy)^5 = x^5y^5$$

$$(-p)^5 = (-1 \times p)^5 = (-1)^5\ p^5 = -p^5$$

$$(x^2y)^5 = (x^2)^5\ y^5 = x^{10}y^5$$

$$(2y^3)^8 = 2^8(y^3)^8 = 256\ y^{24}$$

Law 7:

$$\boxed{\begin{array}{ll} & (x \div y)^m = x^m \div y^m \\ \text{or} & \left(\frac{x}{y}\right)^m = \frac{x^m}{y^m} \end{array}}$$

Rationale:

$$\text{m factors of } \frac{x}{y} = \frac{\text{m factors of x}}{\text{m factors of y}}$$

by rule for multiplication of fractions.

Examples:

$$\left(\frac{a}{b}\right)^7 = \frac{a^7}{b^7}$$

$$\left(-\frac{2x}{3}\right)^3 = \left(\frac{-2x}{3}\right)^3 = \frac{(-2x)^3}{3^3} = \frac{(-2)^3x^3}{3^3} = \frac{-8x^3}{27}$$

or $$\left(-\frac{2x}{3}\right)^3 = \left((-1)\frac{2x}{3}\right)^3 = (-1)^3\left(\frac{2x}{3}\right)^3 = (-1)\frac{(2x)^3}{3^3} = -\frac{8x^3}{27}$$

Law 8:

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x}}$$

Rationale:

By law 3, $(a^{\frac{1}{n}})^n = a^{\frac{n}{n}} = a^1 = a$

Thus n factors of $a^{\frac{1}{n}}$ equals a.

But, by definition, $\sqrt[n]{a}$ is that number, n factors of which equals a

$\therefore \quad a^{\frac{1}{n}} = \sqrt[n]{a}$

Examples:

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

$$(-125)^{\frac{1}{3}} = \sqrt[3]{-125} = -5$$

$$(27x^6)^{\frac{1}{3}} = 27^{\frac{1}{3}}(x^6)^{\frac{1}{3}} = \sqrt[3]{27}\,x^2 = 3x^2$$

Law 9:

$$\boxed{x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m}$$

Rationale:

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} \qquad \text{(law 3)}$$

$$= \sqrt[n]{x^m} \qquad \text{(law 8)}$$

$$\text{But } x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m \qquad \text{(law 3)}$$

$$= (\sqrt[n]{x})^m \qquad \text{(law 8)}$$

<u>Examples:</u>

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$$

$$\text{or } \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

$$(-32x^{10})^{.6} = (-32x^{10})^{\frac{3}{5}} = (-32)^{\frac{3}{5}}(x^{10})^{\frac{3}{5}} = (\sqrt[5]{-32})^3 x^6$$

$$= (-2)^3 x^6 = -8x^6$$

c) <u>Additional Examples of Use of Power Laws</u>

<u>Example 1:</u>

$$(3xy^2)^3 \div (6x^6y^4)$$

$$= \frac{(3xy^2)^3}{6x^6y^4}$$

$$= \frac{3^3x^3(y^2)^3}{6x^6y^4} \qquad \text{(law 6 on numerator)}$$

$$= \frac{27x^3y^6}{6x^6y^4} \qquad \text{(complete exponentiation, law 3)}$$

$= \frac{9}{2} x^{3-6} \; y^{6-4}$ (apply law 2 to x's, y's separately)

$= 4.5 \; x^{-3} y^2$

Example 2:

$$\frac{(-x)^2 (-x^2)}{(-x)^{-2} (-x^{-2})}$$

$$= \frac{(-x)^2 \cancel{(-1)} x^2}{(-x)^{-2} \cancel{(-1)} x^{-2}}$$ (show $-x^2$ as $(-1)x^2$ to separate numerical coefficient (-1) from base x)

$= (-x)^{2-(-2)} \; x^{2-(-2)}$ (law 2 for each base)

$= (-x)^4 x^4$

$= x^4 x^4$ (even no. negative factors yields positive result)

$= x^8$ (law 1)

Example 3:

$$(-\frac{1}{32})^{-.4}$$

$$= (\frac{1}{-32})^{-2/5}$$

$$= \frac{1^{-2/5}}{(-32)^{-2/5}}$$ (law 7)

$$= \frac{1}{(-32)^{-2/5}}$$ ($1^x = 1$ for any x value)

$= (-32)^{2/5}$ (law 4)

$= \left(\sqrt[5]{-32} \right)^2$ (law 9)

$= (-2)^2$ ($\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$)

$= 4$

IV The Four Basic Operations with Algebraic Terms

a) Definitions:

An *algebraic term* is a group of numbers and/or letters associated by multiplication or division only, and separated from other terms by addition or subtraction,

eg, $3x^2$, $-5xy$, 16, xypg are terms

Like terms are terms having identical letter combinations, <u>including exponents</u>,

eg, x, 3x, -17x and

xy^2, $5xy^2$, $-4xy^2$ are groups of like terms,

but 5xy and $-4xy^2$ are not like terms since

the exponent on y differs.

The *numerical coefficient* of a term is the number which multiplies the letter combination,

eg, 3xy, πq^2, -15tsw

↑ ↑ ↑

numerical coefficients

b) Addition and Subtraction of Terms

Like terms ONLY are added/subtracted by adding/subtracting their numerical coefficients. The process of adding/subtracting like terms to simplify an algebraic expression is called *collecting terms*.

Example 1:

$3x^2 + 5x^2 = (3 + 5)x^2$ (Note that letter combination does not change)

$= 8x^2$

Example 2:

$$-15yp^2 + 9yp^2 = (-15 + 9)yp^2$$
$$= -6yp^2$$

Example 3:

$$5qr - 3qr = (5 - 3)qr$$
$$= 2qr$$

Example 4:

$$-15x^2y - 2x^2y = (-15 - 2)x^2y$$
$$= -17x^2y$$

Example 5:

Simplify $-15x^2 + 4xy - y^2 + 2x^2 - 3y^2$

Solution:

$$-15x^2 + 4xy - y^2 + 2x^2 - 3y^2$$

$$= (-15 + 2)x^2 + 4xy + (-1 + (-3))y^2$$ (collect like terms)

$$= -13x^2 + 4xy - 4y^2$$

c) Multiplication and Division of Terms:

Terms are multiplied/divided by multiplying/dividing first the numerical coefficients, then each group of *like powers* (same bases) successively.

Example 1:

$$(5x^2y)(1.3xy^3)$$

$$= (5 \times 1.3)(x^2\ x)(y\ y^3)$$ (group like powers)

$$= 6.5\ x^3y^4$$

Example 2:

$$(-4pq^2)(-3qr^3)$$

$$= (-4 \times (-3))(p)(q^2\ q)(r^3)$$

$$= 12pq^3r^3$$

Example 3:

$$(15x^6) \div (3x^2)$$

$$= \left(\frac{15}{3}\right)\left(\frac{x^6}{x^2}\right)$$

$$= 5x^4$$

Example 4:

$$\frac{120p^2q}{-15q^3r^2} = \frac{120}{-15}\left(\frac{p^2}{1}\right)\left(\frac{q}{q^3}\right)\left(\frac{1}{r^2}\right)$$

$$= -8p^2q^{-2}r^{-2}$$

(Imagine factors of $r^0 = 1$ in the numerator, and $p^0 = 1$ in the denominator if this helps.)

V Multiplication and Division of Polynomials

Definitions:

Monomials, *binomials* and *polynomials* are algebraic expressions having one, two and several *terms*, respectively.

a) Multiplying Binomials by Monomials

Multiply each term of the binomial by the monomial.

Example 1:

terms

$$\underset{\text{Monomial}}{a}\underbrace{(b + c)}_{\text{Binomial}} = ab + ac$$

Example 2:

$$5x(2x - y)$$

$$= 5x(2x + (-y))$$ (Optional step: express binomial as sum of 2 terms.)

$$= 5x(2x) + 5x(-y)$$

$$= 10x^2 + (-5xy)$$

$$= 10x^2 - 5xy$$

b) Multiplying Two Binomials

Multiply each term of second binomial by each term of the first binomial.

Example 1:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Example 2:

$$(2x + y)(x - 5y)$$

$$= 2x(x) + 2x(-5y) + y(x) + y(-5y)$$

$$= 2x^2 - 10xy + xy - 5y^2$$

$$= 2x^2 - 9xy - 5y^2$$ (collect terms in xy)

c) Dividing Binomials by Monomials

Divide each term of the binomial by the monomial.

Example 1:

$$\frac{12x^2 + 4xy}{2x}$$

$$= \frac{12x^2}{2x} + \frac{4xy}{2x}$$ (problem reduces to dividing terms)

$$= 6x + 2y$$

Example 2:

$$\left.\begin{aligned} & -\frac{10x^2 - 4y^2}{2xy} \\ & = -\left(\frac{10x^2}{2xy} - \frac{4y^2}{2xy}\right) \end{aligned}\right\}$$ Note that minus sign in front of quotient applies to entire expression, hence the brackets

$$= -\left(\frac{5x}{y} - \frac{2y}{x}\right)$$

$$= -\frac{5x}{y} + \frac{2y}{x} \quad \underline{\text{or}} \quad -5xy^{-1} + 2x^{-1}y$$

d) Generalizations to Polynomials

To multiply two polynomials, multiply each term of the first by each and every term of the second polynomial.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

VI Simplification of Algebraic Expressions

The order of operations, "BEDMAS" (see 421.10-1, V), holds for simplifying algebraic expressions just as for arithmetic expressions. The following examples illustrate the preceding rules for operations on powers, terms, and polynomials.

Example:

Simplify the following:

(a) $x^2 \div x + x^3 \div x^2 + x$

(b) $aba + aab$

(c) $4x - 7y - (3x - 4y) + x + 3y$

(d) $abc - abc\ (-2)(-\frac{1}{2})(-3)$

(e) $\dfrac{-6ab - 12a^2}{-3a} - \dfrac{3b^2 - 6ab}{-3b}$

Solutions:

(a) $x^2 \div x + x^3 \div x^2 + x$

$= x + x + x$ (÷ precedes +)

$= 3x$ (collect terms)

(b) $aba + aab$

$= aab + aab$ (order of a's, b's does not affect value of product)

$= a^2b + a^2b$

$= 2a^2b$ (collect terms)

(c) $4x - 7y - (3x - 4y) + x + 3y$

$= 4x - 7y - 3x + 4y + x + 3y$ (remove brackets preceded by minus sign by changing sign of <u>all</u> enclosed terms)

$= (4 - 3 + 1)x + (-7 + 4 + 3)y$ (collect like terms)

$= 2x$

(d) $abc - abc\ (-2)\ (-\frac{1}{2})(-3)$

$= abc - abc\ (-3)$ (3 negative factors (odd no.) give negative product)

$= abc + 3abc$ (to subtract, add the opposite)

$= 4abc$

(e) $\dfrac{-6ab - 12a^2}{-3a} - \dfrac{3b^2 - 6ab}{-3b}$

$= \dfrac{-6ab}{-3a} + \dfrac{-12a^2}{-3a} - (\dfrac{3b^2}{-3b} + \dfrac{-6ab}{-3b})$ (Express binomials as sum of 2 terms)

$= 2b + 4a - (-b + 2a)$

$= 2b + 4a + b - 2a$ (remove brackets)

$= 2a + 3b$ (collect terms

<u>NB</u> Brackets preceded by a "+" may be inserted or removed without altering enclosed terms, but brackets preceded by a "-" may be inserted or removed only by altering signs of <u>all</u> terms enclosed.

Assignment

1. If a = 12, b = 2 and c = -3, evaluate the following:

   (a) $-a + \frac{5b}{6} + \frac{cb}{a}$

   (b) $a + 2a - 3c^2$

   (c) $6b^2 - a - b^2 + c$

2. Simplify

   (a) a^4a^6 (b) $\frac{1}{2}a(\frac{1}{4}a^7)(\frac{3}{4}a^2)$

   (c) $b^3b^4b^5$ (d) $3 \times 3^2 \times 3^4$

   (e) $m^7 \cdot m^4 \div m^5$ (f) $a^6 \div a^{-5} \cdot a^8$

   (g) $\frac{a^7}{a^5} \cdot \frac{a}{a^4}$ (h) $\frac{b^6b^4}{b^3}$

   (i) $(a^7)^2$ (j) $(-3a^2)^3$

   (k) $(\frac{1}{2}x^4)^5$ (l) $\sqrt[3]{a}\ \sqrt{a}$

   (m) $\sqrt{\sqrt[3]{a}}$ (n) $(-3xy^{\frac{1}{2}})^2$

   (o) $x^6x^{-2}x^{-4}$ (p) $(\frac{\sqrt{x}}{y^2})^2$

3. Evaluate:

   (a) $3 \cdot \sqrt{3} \cdot \sqrt{3^3}$ (b) $\left(\frac{1}{4}\right)^{2 \cdot 5}$ (c) $(16)^{-0 \cdot 25}$

   (d) $\left(\frac{2^2}{3^3}\right)^{-1}$ (e) $(-3)^{-3}$ (f) $36^{1/2}$

   (g) $\left(-\frac{1}{32}\right)^{-1/5}$ (h) $(-8)^{5/3}$ (i) $\left(-\frac{27}{64}\right)^{-2/3}$

4. Write each expression without negative or zero exponents and simplify.

(a) $\dfrac{3a^0 - b^0}{a^0 + (3b)^0}$ (b) $(16\ x^{16})^{1/2}$

(c) $-(-3a^{0.4}\ b^{0.6})^5$ (d) $\dfrac{3x^{-3}\ y^2}{6^{-1}\ z^{-2}}$

5. The mass of an electron is 0.00055 a.m.u. and 1 a.m.u. is 1.66×10^{-24} g. Calculate the mass of the electron in grams.

6. It requires 3.1×10^{10} fissions per second to produce 1 watt of energy. How many fissions per second are required to produce 200 Megawatts.

7. Simplify:

(a) $2a + 3a + 6a$

(b) $5x^2 + 2x + 3 + 5xy + 4x + 2 + x^2$

(c) $5x + 12y + 20x + 8y$

(d) $2c + 8a + 6c + 2b + 3b + 4c$

(e) $3j + k - 4j + 11k - 7k - j$

(f) $a + a + a + a - 5a + 11a$

(g) $x + 3xy^2 + 2x + y - 3x + y^2x$

(h) $x^2y^3 + 3x^2y^3 + 1 - x$

(i) $x + y + z - 2y + 3z - 6x$

8. Simplify:

(a) $\frac{-6y^2}{3y}$ (b) $\frac{7x}{21xy}$

(c) $\frac{15ab}{3a}$ (d) $-\frac{30pq}{15pq}$

(e) $\frac{-2x^2}{-2x}$ (f) $\frac{-27ab^2c}{9b}$

(g) $\frac{6y}{3x}$ (h) $\frac{x^2yz}{2yz}$

(i) $(6x^2y)(-4y^3p)$ (j) $(-11pq)(-2ps^2t)$

9. Simplify:

(a) $(x + 4y)(x - 8y)$ (b) $(3x + 2)(5x + 4)$

(c) $(3a - 2c)(4a - 5c)$ (d) $(x^2 - y)(y + x^2)$

(e) $\frac{6x^2 - 2x}{-2x} - \frac{9x - 3}{-3}$ (f) $\frac{4x + 10y}{2}$

(g) $\frac{-10a^2 - 5}{-5} - \frac{-3a^2 - 6}{-3}$ (h) $\frac{8x^2 + 10x}{2x}$

(i) $\frac{3x^2 - 15x}{3x} - \frac{12x - 18}{-6}$ (j) $\frac{14x^2 + 21x}{7x} - \frac{3x^2 + 9x}{3}$

10. Simplify:

(a) $(8mn)(4mx)$

(b) $(9abc)(-4bcd)$

(c) $-3y(6m - 5t)$

(d) $5(4h - 6k)$

(e) $-3(x + y) + 10(2x - 3y) + 5(2y - 3x)$

(f) $2x(x + y) - (x^2 + xy) + (x - x)$

(g) $8c + 3k - (5c + 2k)$

10. Cont'd

(h) $2b - c + (8c - 4b) + b$

(i) $3a - 5x - (4a + x) - 2a$

(j) $ab + ab^2 \div b$

(k) $xy + x^2y^2 \div xy$

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